THE STABILITY OF MACLAURIN ELLIPSOIDS OF A ROTATING FLUID

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The motion of a fluid under the action of the forces of mutual attraction of its particles in accordance with Newton's law for the case in which the displacements are expressed as linear functions of the coordinates was first investigated by Dirichlet. He showed that for known initial conditions the fluid can move so that its free surface during the time of the motion remains the surface of an ellipsoid whose axes, generally speaking, change their magnitudes and directions with the passage of time. These investigations were continued by Dedekind, Riemann, Steklov and by a number of other authors (see the bibliography in Lamb [1]). It has been shown, in particular, that in this case there can be rotations of the whole fluid as a single solid body about the smallest axis of a tri-axial ellipsoid (the Jacobi ellipsoid) or about the polar axis of an oblate ellipsoid (the Maclaurin ellipsoid), the existence of which in the general case was established considerably before the publication of the work of Dirichlet (1860).

The question of the stability of the ellipsoidal figures of equilibrium of a rotating fluid attracted the steady attention of many investigators, starting with Liouville and Riemann.

Riemann [2] investigated the stability of Maclaurin and Jacobi ellipsoids with respect to initial displacements and velocities which satisfy the hypotheses of Dirichlet. Noting the analogy between the differential equations which define under some additional special assumptions the semi-axes of a fluid ellipsoid as a function of time and the differential equations of the motion of a material point on some surface under the action of forces which possess a force potential, Riemann used a theorem of Lagrange concerning the minimum of this force potential as a criterion for the stability of figures of equilibrium. Thus, he established that Jacobi ellipsoids are always stable and that Maclaurin ellipsoids are stable or unstable according to whether their eccentricities are less than or greater than 0.9528... In this connection, as is not difficult to see, by Riemann stability we mean stability with respect to the lengths of the fluid ellipsoid semi-axes and with respect to the rates of variation, apart from the Dirichlet condition that in the disturbed motion the moment of momentum and the vorticity have the same values as in the case of the figures of equilibrium.

Thomson and Tait in their treatise [3] indicate (without proof) that all planetary ellipsoids of revolution are stable if the fluid for all time remains an ellipsoid of revolution. But if the condition that the fluid always keep the form of an ellipsoid is imposed, then Maclaurin ellipsoids are stable or unstable depending on whether their eccentricities are less than or greater than 0.8126..., and tri-axial ellipsoids are always stable.

A rigorous definition of the stability of figures of equilibrium of a fluid as the stability of its form was first given by Liapunov [4]; the theory, which he proved and which is a generalization of a theory of Raus, gives a sufficient condition for the stability of the form of equilibrium for a given moment of momentum of the fluid.

Using this criterion Liapunov proved that ellipsoids of revolution are stable as long as their eccentricities remain less than 0.8126..., and that tri-axial ellipsoids are stable within certain narrow limits; the Jacobi ellipsoid of revolution is stable. For the particular case in which the surface of the fluid remains ellipsoidal, the upper limit of the eccentricities of stable Maclaurin ellipsoids remains just the same as in the general case, but Jacobi ellipsoids in this case are always stable.

Thus, if we confine ourselves to the case of ellipsoidal disturbances, then the conclusions of Riemann, Thomson and Tait and Liapunov with respect to the stability of Jacobi ellipsoids coincide, but with respect to the stability of Maclaurin ellipsoids they differ.

In this connection the question arises: is it not possible to consider the problem of the stability of Maclaurin ellipsoids from some other point of view which differs from the one presented, and what will the results be? It is especially tempting to try to solve this problem of the stability in the sense of Liapunov and by the methods of the stability theory of Liapunov for a system with a finite number of degrees of freedom.

The solution of this problem for the condition that the initial disturbances satisfy the Dirichlet hypotheses is given below.

1. We will investigate an ideal homogeneous incompressible fluid, the particles of which are attracted to each other in accordance with Newton's law, while the pressure on its free surface remains constant.

For the indicated conditions the mass-center of the fluid moves uniformly and rectilinearly; without loss of generality we will consider it to be stationary. We will take the mass-center for the origin of two rectangular systems of coordinates: a stationary system $Ox_1y_1z_1$ and a moving system $Oxyz$, which moves coupled with the fluid about its mass-center. We will denote by p, q, r the projections on the x-, y-, z-axes of the instantaneous angular velocity ω of the moving system of coordinates relative to the stationary system.

We shall write the Eulerian form of the equations of motion of the fluid in the moving axes:

$$\begin{aligned}
\frac{dv_x}{dt} + qv_z - rv_y &= -\frac{\partial U}{\partial x} - \frac{1}{\rho}\frac{\partial p_1}{\partial x}\\
\frac{dv_y}{dt} + rv_x - pv_z &= -\frac{\partial U}{\partial y} - \frac{1}{\rho}\frac{\partial p_1}{\partial y}\\
\frac{dv_z}{dt} + pv_y - qv_x &= -\frac{\partial U}{\partial z} - \frac{1}{\rho}\frac{\partial p_1}{\partial z}\\
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &+ \frac{\partial v_z}{\partial z} = 0
\end{aligned} \tag{1.1}$$

Here v_x, v_y, v_z denote the projections on the moving axes of the vector $\mathbf{v}$, the fluid velocity relative to the coordinate system $Ox_1y_1z_1$, ρ is the density of the fluid, p_1 is the hydrodynamic pressure, and U is the attraction potential.

We will confine the investigation only to such motions of the fluid for which its free surface remains for all time an ellipsoid [2,5]

$$F(x, y, z, t) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \tag{1.2}$$

with variable axes $a(t)$, $b(t)$, $c(t)$, and we will assume that

$$v_x = \frac{\partial \varphi}{\partial x} + \omega_2 z - \omega_3 y, \qquad v_y = \frac{\partial \varphi}{\partial y} + \omega_3 x - \omega_1 z, \qquad v_z = \frac{\partial \varphi}{\partial z} + \omega_1 y - \omega_2 x \tag{1.3}$$

Here $\omega_i (i = 1,2,3)$ is a function only of time t, and $\phi(x, y, z, t)$ is a harmonic function of the coordinates in a region τ, the bounding surface (1.2).

We obtain the boundary condition for the function $\phi(x, y, z, t)$ from the kinetic condition for the free surface

$$\frac{\partial F}{\partial x} u + \frac{\partial F}{\partial y} v + \frac{\partial F}{\partial z} w + \frac{\partial F}{\partial t} = 0$$

where u, v, w denote projections of the fluid velocity relative to the coordinate system $Oxyz$ on the axes of the latter. This condition taking into account equations (1.3) takes the following form on the surface (1.2):

$$\frac{\partial\varphi}{\partial x}\frac{x}{a^2}+\frac{\partial\varphi}{\partial y}\frac{y}{b^2}+\frac{\partial\varphi}{\partial z}\frac{z}{c^2}=\frac{x^2}{a^3}a'+\frac{y^2}{b^3}b'+\frac{z^2}{c^3}c'+$$
$$+(\omega_1-p)\frac{c^2-b^2}{b^2c^2}yz+(\omega_2-q)\frac{a^2-c^2}{a^2c^2}xz+(\omega_3-r)\frac{b^2-a^2}{a^2b^2}xy \tag{1.4}$$

where for brevity we introduce the notation

$$a'=\frac{da}{dt},\qquad b'=\frac{db}{dt},\qquad c'=\frac{dc}{dt}$$

It is easy to see that the harmonic function

$$\varphi(x,y,z,t)=\frac{1}{2}\left(\frac{a'}{a}x^2+\frac{b'}{b}y^2+\frac{c'}{c}z^2\right)+\frac{b^2-c^2}{b^2+c^2}(p-\omega_1)yz+$$
$$+\frac{c^2-a^2}{c^2+a^2}(q-\omega_2)xz+\frac{a^2-b^2}{a^2+b^2}(r-\omega_3)xy \tag{1.5}$$

satisfies condition (1.4); at the same time $a(t)$, $b(t)$, $c(t)$ must satisfy the equation

$$\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}=0 \tag{1.6}$$

which appears as a result of the incompressibility equation.

Taking into account (1.5) the equalities (1.3) take the following form:

$$\begin{aligned}
v_x&=\frac{a'}{a}x+\frac{(a^2-b^2)r-2a^2\omega_3}{a^2+b^2}y+\frac{(c^2-a^2)q+2a^2\omega_2}{a^2+c^2}z\\
v_y&=\frac{b'}{b}y+\frac{(b^2-c^2)p-2b^2\omega_1}{b^2+c^2}z+\frac{(a^2-b^2)r+2b^2\omega_3}{a^2+b^2}x\\
v_z&=\frac{c'}{c}z+\frac{(c^2-a^2)q-2c^2\omega_2}{c^2+a^2}x+\frac{(b^2-c^2)p+2c^2\omega_1}{c^2+b^2}y
\end{aligned} \tag{1.7}$$

To determine the functions $\omega_i(t)$ we will make use of the Helmholtz vortex equation

$$\frac{d\Omega}{dt}+\omega\times\Omega=(\Omega\nabla)\mathbf{v}$$

where $\Omega=\operatorname{rot}\mathbf{v}$ in the case under consideration has the following projections on the moving axes:

$$\Omega_x=2\omega_1,\qquad \Omega_y=2\omega_2,\qquad \Omega_z=2\omega_3$$

Taking into account formulas (1.7), we write the Helmholtz equations in the moving axes in the form

$$\begin{aligned}
&\frac{d}{dt}\frac{\omega_1}{a}-\frac{2a}{a^2+b^2}r\omega_2+\frac{2a}{a^2+c^2}q\omega_3+\frac{2a(c^2-b^2)}{(a^2+c^2)(a^2+b^2)}\omega_2\omega_3=0\\
&\frac{d}{dt}\frac{\omega_2}{b}-\frac{2b}{b^2+c^2}p\omega_3+\frac{2b}{a^2+b^2}r\omega_1+\frac{2b(a^2-c^2)}{(a^2+b^2)(b^2+c^2)}\omega_3\omega_1=0\\
&\frac{d}{dt}\frac{\omega_3}{c}-\frac{2c}{c^2+a^2}q\omega_1+\frac{2c}{b^2+c^2}p\omega_2+\frac{2c(b^2-a^2)}{(b^2+c^2)(c^2+a^2)}\omega_1\omega_2=0
\end{aligned} \tag{1.8}$$

To construct differential equations for $p(t)$, $q(t)$, $r(t)$ we will make use of a theory about the moment of momentum of the system, according to which we have

$$\frac{dG_x}{dt}+qG_z-rG_y=0,\quad \frac{dG_y}{dt}+rG_x-pG_z=0\quad \frac{dG_z}{dt}+pG_y-qG_x=0 \tag{1.9}$$

Here G_x, G_y, G_z denote the projections on the coordinate axes x, y, z of the moment of momentum of the fluid mass relative to the point O.

Taking into account the equalities (1.7), it is easy to find

$$G_x=A_1p+A_2\omega_1,\qquad G_y=B_1q+B_2\omega_2,\qquad G_z=C_1r+C_2\omega_3 \tag{1.10}$$

where for brevity we introduce the following notation:

$$\begin{aligned} A_1&=\frac{M}{5}\frac{(b^2-c^2)^2}{b^2+c^2}, & B_1&=\frac{M}{5}\frac{(c^2-a^2)^2}{c^2+a^2}, & C_1&=\frac{M}{5}\frac{(a^2-b^2)^2}{a^2+b^2}\\ A_2&=\frac{4M}{5}\frac{b^2c^2}{b^2+c^2}, & B_2&=\frac{4M}{5}\frac{c^2a^2}{c^2+a^2}, & C_2&=\frac{4M}{5}\frac{a^2b^2}{a^2+b^2}\end{aligned} \tag{1.11}$$

and $M=4/3\,\pi\rho\,abc$ is the fluid mass.

We will finally formulate the differential equations for $a(t)$, $b(t)$, $c(t)$. It is easy to see [5] that in the case under consideration the attraction potential for the interior points is

$$U=\tfrac{1}{2}f\,(Px^2+Qy^2+Rz^2)-fH$$

where

$$\begin{gathered} H=\tfrac{3}{4}M\int_0^\infty\frac{d\lambda}{\sqrt{\varphi(\lambda)}},\qquad \varphi(\lambda)=(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\\ P=-\frac{2}{a}\frac{\partial H}{\partial a},\qquad Q=-\frac{2}{b}\frac{\partial H}{\partial b},\qquad R=-\frac{2}{c}\frac{\partial H}{\partial c}\end{gathered} \tag{1.12}$$

f is the constant of attraction; without loss of generality we will further consider $f=1$.

Substituting in equations (1.1) the right-hand side of the equalities (1.7) for v_x, v_y, v_z and taking into account equations (1.8) and (1.9), and also (1.12), we obtain

$$(P+w_x)\,x+\frac{1}{\rho}\frac{\partial p_1}{\partial x}=0,\quad (Q+w_y)\,y+\frac{1}{\rho}\frac{\partial p_1}{\partial y}=0\quad (R+w_y)z+\frac{1}{\rho}\frac{\partial p_1}{\partial z}=0 \tag{1.13}$$

where w_x, w_y, w_z are quantities which do not depend on the coordinates of the fluid particles:

$$w_x = \frac{a''}{a} - \frac{a^2-c^2}{(a^2+c^2)^2}(a^2+3c^2)q^2 - \frac{a^2-b^2}{(a^2+b^2)^2}(a^2+3b^2)r^2 + \tag{1.14}$$
$$+ \frac{4(a^2-c^2)c^2}{(a^2+c^2)^2}q\omega_2 + \frac{4(a^2-b^2)b^2}{(a^2+b^2)^2}r\omega_3 - \frac{4a^2b^2}{(a^2+b^2)^2}\omega_3{}^2 - \frac{4a^2c^2}{(a^2+c^2)^2}\omega_2{}^2$$

$$w_y = \frac{b''}{b} - \frac{b^2-a^2}{(a^2+b^2)^2}(b^2+3a^2)r^2 - \frac{b^2-c^2}{(b^2+c^2)^2}(b^2+3c^2)p^2 +$$
$$+ \frac{4(b^2-a^2)a^2}{(a^2+b^2)^2}r\omega_3 + \frac{4(b^2-c^2)c^2}{(b^2+c^2)^2}p\omega_1 - \frac{4b^2c^2}{(b^2+c^2)^2}\omega_1{}^2 - \frac{4a^2b^2}{(a^2+b^2)^2}\omega_3{}^2$$

$$w_z = \frac{c''}{c} - \frac{c^2-b^2}{(c^2+b^2)^2}(c^2+3b^2)p^2 - \frac{c^2-a^2}{(c^2+a^2)^2}(c^2+3a^2)q^2 + \frac{4(c^2-b^2)b^2}{(c^2+b^2)^2}p\omega_1 +$$
$$+ \frac{4(c^2-a^2)a^2}{(c^2+a^2)^2}q\omega_2 - \frac{4c^2a^2}{(c^2+a^2)^2}\omega_2{}^2 - \frac{4b^2c^2}{(b^2+c^2)^2}\omega_1{}^2$$

Integrating equations (1.13), we find

$$\frac{1}{2}[(P+w_x)x^2 + (Q+w_y)y^2 + (R+w_z)z^2] + \frac{p_1-p_0}{\rho} = \sigma(t)$$

where $\sigma(t)$ is an arbitrary function of time.

But on the free surface of the fluid the pressure p_0, according to the condition specified, is constant, therefore, in order that this surface have the form of an ellipsoid (1.2) it is necessary and sufficient that an ellipsoid be coincident with a surface of constant pressure. Consequently, the function $\sigma(t)$ must be determined so that the surface of constant pressure coincides with the surface (1.2). Comparing coefficients, we obtain

$$(P+w_x)a^2 = (Q+w_y)b^2 = (R+w_z)c^2 = 2\sigma(t) \tag{1.15}$$

In addition, the hydrodynamic pressure will be determined by the formula

$$\frac{p_1-p_0}{\rho} = \sigma(t)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right) \tag{1.16}$$

Hence it follows that the function $\sigma(t)$ must not take negative values.

From the relations (1.15) we find equations for $a(t)$, $b(t)$, $c(t)$:

$$w_x = \frac{2\sigma}{a^2} - P, \qquad w_y = \frac{2\sigma}{b^2} - Q, \qquad w_z = \frac{2\sigma}{c^2} - R \tag{1.17}$$

into the left-hand sides of which we must substitute in place of w_x, w_y, w_z the expressions according to (1.14).

Thus, the problem of studying the motion of a fluid mass which has the form of an ellipsoid (1.2) with varying axes is reduced to the investigation of the ten equations (1.17), (1.6), (1.8) and (1.9) with the same number of unknowns a, b, c, ω_1, ω_2, ω_3, p, q, r, σ.

This system of equations permits a series of first integrals.

We multiply equations (1.17) by $a'a$, $b'b$, $c'c$ respectively and add them, we multiply the result by $1/5\,\rho\,d\tau$ and integrate over the whole volume of the fluid; we add the resulting equation to the sum of the products of equations (1.9) by p, q, r respectively, whence taking into account equations (1.6) and (1.8), we obtain the energy integral

$$\frac{M}{10}(a'^2+b'^2+c'^2)+\tfrac{1}{2}(A_1p^2+B_1q^2+C_1r^2+A_2\omega_1^2+B_2\omega_2^2+C_2\omega_3^2)+ \\ +W=\text{const} \tag{1.18}$$

where the potential energy of the system is

$$W=\tfrac{1}{2}\,\rho\int_\tau U d\tau=-\tfrac{2}{5}MH$$

Multiplying equations (1.9) by G_x, G_y, G_z respectively and adding, we obtain an equation from which there follows immediately the integral expressing the constancy of the moment of momentum of the system

$$(A_1p+A_2\omega_1)^2+(B_1q+B_2\omega_2)^2+(C_1r+C_2\omega_3)^2=\text{const} \tag{1.19}$$

We now multiply equations (1.8) by ω_1/a, ω_2/b, ω_3/c respectively and add, we easily obtain the integral expressing the constancy of the vorticity

$$\left(\frac{\omega_1}{a}\right)^2+\left(\frac{\omega_2}{b}\right)^2+\left(\frac{\omega_3}{c}\right)^2=\text{const} \tag{1.20}$$

Finally, multiplying equation (1.6) by abc, we obtain the integral expressing the constancy of the mass of the fluid

$$abc=\text{const} \tag{1.21}$$

2. The system of equations of motion of the fluid mass permits the particular solution

$$a=a_0,\quad b=b_0,\quad c=c_0,\quad a'=b'=c'=0 \\ p=q=0,\quad r=\omega,\quad \omega_1=\omega_2=0,\quad \omega_3=\omega,\ \sigma=\sigma_0 \tag{2.1}$$

which describes a uniform rotation of all the fluid as a single solid body about the axis Oz with angular velocity ω. The constants a, b, c, ω, σ must here satisfy equations (1.17) which take the form

$$-\omega^2=\frac{2\sigma_0}{a_0^2}-P,\qquad -\omega^2=\frac{2\sigma_0}{b_0^2}-Q,\qquad O=\frac{2\sigma_0}{c_0^2}-R$$

Hence we obtain

$$(P-\omega^2)\,a_0{}^2=(Q-\omega^2)\,b_0{}^2=Rc_0{}^2 \tag{2.2}$$

Investigation of these equations leads, as is well-known [6], to the following conclusions: figures of equilibrium of a rotating fluid exist which have the form of ellipsoids of revolution (Maclaurin ellipsoids) when $a_0 = b_0 > c_0$ and of tri-axial ellipsoids (Jacobi ellipsoids) when the axis c is the smallest axis of the ellipsoid (1.2).

If $0 < 1/2(\omega^2/\pi f\rho) < 0.225\ldots$, two Maclaurin ellipsoids which differ in oblateness from each other correspond to each value of ω. For $1/2(\omega^2/\pi f\rho) = 0.225\ldots$ both ellipsoids of revolution coincide, reducing to one limiting Maclaurin ellipsoid. For $1/2(\omega^2/\pi f\rho) > 0.225\ldots$ ellipsoidal figures of equilibrium of a rotating fluid do not exist.

In the case of tri-axial ellipsoids, if $0 < 1/2(\omega^2/\pi f\rho) < 0.1871$, for each value of ω there correspond two identical Jacobi ellipsoids in which only the x- and y-axes are transposed. For $1/2(\omega^2/\pi f\rho) = 0.1871$ the axes a_0 and b_0 become equal and the Jacobi ellipsoid turns into an ellipsoid of revolution E, which at the same time is also a Maclaurin ellipsoid. For $1/2(\omega^2/\pi f\rho) > 0.1871$ tri-axial ellipsoids of equilibrium of a rotating fluid do not exist.

The ellipsoid E, which belongs simultaneously to two series of figures of equilibrium, is a bifurcated ellipsoid.

We shall pass now to the investigation of the stability of Maclaurin ellipsoids, restricting consideration only to disturbances which satisfy the Dirichlet hypotheses under which the free surface of the fluid remains an ellipsoid (1.2).

It is natural to refer to such disturbances as ellipsoidal disturbances [5]; for information on their figure of equilibrium the resulting motion of the fluid will be described by equations (1.17), (1.6), (1.8) and (1.9).

For the stability of the figures of equilibrium, we will understand stability in the sense of Liapunov with respect to the variables a, b, c, a', b', c', ω_1, ω_2, ω_3, p, q, r.

And so we will put $a_0 = b_0$ and we will assume for the undisturbed motion the particular solution (2.1) of the equations of motion. In the disturbed motion we will put

$$a=a_0+\alpha,\quad b=a_0+\beta,\quad c=c_0+\gamma,\quad r=\omega+\xi,\quad \omega_3=\omega+\eta$$

and for the remaining variables we will keep the previous notation. Substituting these quantities in equations (1.17), (1.6), (1.8), (1.9), we obtain a system of equations for the disturbed fluid; we will not write down the latter equations explicitly. It is evident that the exact equations of the disturbed motion permit the following first integrals which correspond to the integrals (1.18)-(1.21):

$$V_1 = A_{10}(p^2+q^2) + A_{20}(\omega_1^2+\omega_2^2) + C_{20}\eta^2 + 2C_{20}\omega\eta\left(1+\frac{\alpha+\beta}{a_0}\right) + \\ + \frac{M}{5}\{\alpha'^2+\beta'^2+\gamma'^2+2\omega^2 a_0(\alpha+\beta)+\omega^2(\alpha^2+\beta^2)+2P_0a_0(\alpha+\beta) + \\ + 2R_0c_0\gamma\} + \left(\frac{\partial^2 W}{\partial a^2}\right)_0\alpha^2 + \left(\frac{\partial^2 W}{\partial b^2}\right)_0\beta^2 + \left(\frac{\partial^2 W}{\partial c^2}\right)_0\gamma^2 + 2\left[\left(\frac{\partial^2 W}{\partial b\,\partial c}\right)_0\beta\gamma + \\ + \left(\frac{\partial^2 W}{\partial c\,\partial a}\right)_0\gamma\alpha + \left(\frac{\partial^2 W}{\partial a\,\partial b}\right)_0\alpha\beta\right] + \ldots = \text{const}$$

$$V_2 = A_{10}^2(p^2+q^2) + 2A_{10}A_{20}(p\omega_1+q\omega_2) + A_{20}^2(\omega_1^2+\omega_2^2) + C_{20}^2\eta^2 + \tag{2.3}$$

$$+ \frac{4M}{5}C_0\omega^2[a_0(\alpha+\beta)+\alpha^2+\beta^2+\alpha\beta] + 2C_{20}\omega\eta\left(C_0+2C_0\frac{\alpha+\beta}{a_0}\right) + \ldots = \text{const}$$

$$V_3 = \frac{\omega_1^2}{a_0^2} + \frac{\omega_2^2}{a_0^2} + \frac{2\omega}{c_0^2}\eta - \frac{2\omega^2}{c_0^3}\gamma + \frac{\eta^2}{c_0^2} + \frac{3\omega^2}{c_0^4}\gamma^2 - \frac{4\omega}{c_0^3}\eta\gamma + \ldots = \text{const}$$

$$V_4 = a_0c_0(\alpha+\beta) + a_0^2\gamma + a_0\gamma(\alpha+\beta) + c_0\alpha\beta + \alpha\beta\gamma = 0$$

The dots here and below designate omitted terms of order greater than second; the index o indicates that the corresponding quantities must be computed for values of $a = a_0$, $b = b_0$, $c = c_0$, $C_0 = C_{10} + C_{20}$. We will eliminate the variable γ from the first integrals V_1 = const and V_3 = const, using the integral $V_4 = 0$. Solving the latter equation for γ, we obtain

$$\gamma = -\frac{c_0}{a_0}\left(\alpha+\beta-\frac{\alpha^2+\alpha\beta+\beta^2}{a_0}\right) + \ldots$$

and, substituting in the former, we will have, taking into account equations (2.2),

$$V_1^* = A_{10}(p^2+q^2) + A_{20}(\omega_1^2+\omega_2^2) + C_{20}\eta^2 + 2C_{20}\omega\eta\left(1+\frac{\alpha+\beta}{a_0}\right) + \\ + \frac{M}{5}\left\{\alpha'^2+\beta'^2+\gamma'^2+4\omega^2a_0(\alpha+\beta)+\left(2R_0\frac{c_0^2}{a_0^2}+\omega^2\right)(\alpha^2+\beta^2)\right\} + \\ + \left[\frac{\partial^2 W}{\partial a^2}+\frac{c^2}{a^2}\frac{\partial^2 W}{\partial c^2}-2\frac{c}{a}\frac{\partial^2 W}{\partial c\,\partial a}\right]_0(\alpha^2+\beta^2) + \\ + 2\left[\frac{\partial^2 W}{\partial a\,\partial b}-2\frac{c}{a}\frac{\partial^2 W}{\partial c\,\partial a}+\frac{c^2}{a^2}\frac{\partial^2 W}{\partial c^2}+\frac{M}{5}R\frac{c^2}{a^2}\right]_0\alpha\beta + \ldots = \text{const}$$

$$V_3^* = \frac{\omega_1^2+\omega_2^2}{a_0^2} + \frac{2\omega}{c_0^2}\eta + \frac{2\omega^2}{a_0c_0^2}(\alpha+\beta) + \frac{\eta^2}{c_0^2} + \\ + \frac{4\omega}{a_0c_0^2}\eta(\alpha+\beta) + \frac{\omega^2}{a_0^2c_0^2}(\alpha^2+4\alpha\beta+\beta^2) + \ldots = \text{const}$$

We will consider the function

$$V = V_1^* - \frac{1}{C_0} V_2 + \mu \frac{c_0^4}{4\omega^2} V_3^{*2} =$$

$$= A_{10} \frac{C_0 - A_{10}}{C_0} (p^2 + q^2) - 2A_{10}A_{20} \frac{1}{C_0} (p\omega_1 + q\omega_2) + A_{20} \frac{C_0 - A_{20}}{C_0} (\omega_1^2 + \omega_2^2) +$$
$$+ \frac{M}{5} (\alpha'^2 + \beta'^2 + \gamma'^2) + \left(A_{11} + \mu \frac{\omega^2}{a_0^2}\right)(\alpha^2 + \beta^2) + 2\left(A_{12} + \mu \frac{\omega^2}{a_0^2}\right)\alpha\beta +$$
$$+ 2(\mu - C_{20}) \frac{\omega}{a_0} \eta (\alpha + \beta) + \mu\eta^2 + \dots \qquad (2.4)$$

where

$$A_{11} = \left(\frac{\partial^2 W}{\partial a^2} + \frac{c^2}{a^2} \frac{\partial^2 W}{\partial c^2} - 2\frac{c}{a} \frac{\partial^2 W}{\partial c\, \partial a}\right)_0 + \frac{M}{5}\left(2R_0 \frac{c_0^2}{a_0^2} - 3\omega^2\right)$$
$$A_{12} = \left(\frac{\partial^2 W}{\partial a\, \partial b} - 2\frac{c}{a} \frac{\partial^2 W}{\partial c\, \partial a} + \frac{c^2}{a^2} \frac{\partial^2 W}{\partial c^2}\right)_0 + \frac{M}{5}\left(R_0 \frac{c_0^2}{a_0^2} - 2\omega^2\right) \qquad (2.5)$$

As is seen, the expansion of the function V into a series begins with terms of second order which are quadratic forms of the variables α', β', γ'; p, ω_1, q, ω_2; α, β, η. If the signs of the latter are determined, the sign of the function V will be determined. The first two of them are positive definite taking into consideration that in the case under consideration $a_0 > c_0$. We will find the conditions of positive definiteness for the quadratic forms of the variables α, β, η.

According to the Sylvester criterion

$$A_{11} + \mu \frac{\omega^2}{a_0^2} > 0, \qquad (A_{11} - A_{12})\left(A_{11} + A_{12} + 2\mu \frac{\omega^2}{a_0^2}\right) > 0$$
$$(A_{11} - A_{12})\left[\mu\left(A_{11} + A_{12} + 4C_{20} \frac{\omega^2}{a_0^2}\right) - 2C_{20}^2 \frac{\omega^2}{a_0^2}\right] > 0$$

Obviously, these inequalities can always be satisfied by choosing some positive value of the constant μ, if only the conditions

$$A_{11} + A_{12} + 4C_{20} \frac{\omega^2}{a_0^2} > 0, \qquad A_{11} - A_{12} > 0 \qquad (2.6)$$

are fulfilled.

Taking into account the designations of (1.11) and (2.5) and taking into consideration (1.12) and (2.2) we have

$$A_{11} + A_{12} + 4C_{20} \frac{\omega^2}{a_0^2} =$$
$$= \frac{3M^2}{5a_0^3} \int_0^\infty \left[\frac{2a_0^2}{(a_0^2 + \lambda)^2} + \frac{2c_0^2 (a_0^2 - c_0^2)}{(a_0^2 + \lambda)(c_0^2 + \lambda)^2} + \frac{c_0^2}{(c_0^2 + \lambda)^2}\right] \frac{\lambda\, d\lambda}{(a_0^2 + \lambda)\sqrt{c_0^2 + \lambda}}$$

$$A_{11} - A_{12} = \frac{3M^2}{5a_0^3} \int_0^\infty \left(\frac{c_0^2}{c_0^2 + \lambda} - \frac{a_0^4}{(a_0^2 + \lambda)^2}\right) \frac{d\lambda}{(a_0^2 + \lambda)\sqrt{c_0^2 + \lambda}}$$

Because $a_0 > c_0$ for Maclaurin ellipsoids, then it is obvious that the first of the conditions (2.6) is always satisfied. We will consider the second of these inequalities. Performing the integration and dropping a positive factor, we reduce it to the form

$$l\,[l\,(13 + 3l^2) - (3 + 14l^2 + 3l^4)\operatorname{arc\,ctg} l] > 0$$
$$\left(l = \frac{c_0}{\sqrt{a_0^2 - c_0^2}},\ \varepsilon = \frac{\sqrt{a_0^2 - c_0^2}}{a_0} = \frac{1}{\sqrt{1 + l^2}}\right) \tag{2.7}$$

Here l is a quantity, the inverse of the second eccentricity of the ellipsoid (1.2); ϵ is its first eccentricity.

We will assume [4]

$$u\,(l) \equiv \frac{l\,(13 + 3l^2)}{3 + 14l^2 + 3l^4} - \operatorname{arc\,ctg} l$$

and we find

$$\frac{du}{dl} = \frac{16\,(3 + l^2)\,(1 - l^2)}{(1 + l^2)\,(3 + 14l^2 + 3l^4)^2}$$

Hence it is seen that as l increases from 0 to 1 the function $u(l)$ increases, reaching a maximum for $l = 1$, and for further increases in l constantly decreases; in addition, $u(0) = -\pi/2$, $u(\infty) = 0$. On this basis we conclude that the equation

$$l\,(13 + 3l^2) - (3 + 14l^2 + 3l^4)\operatorname{arc\,ctg} l = 0 \tag{2.8}$$

has only one positive finite root $l_0 < 1$, and that when $l > l_0$ the condition (2.7) is satisfied.

Equation (2.8) is used, as is well-known [4], in determining the Maclaurin ellipsoid with which the limiting Jacobi ellipsoid coincides for $1/2(\omega^2/\pi + \rho) = 0.187$. The eccentricity of this ellipsoid is $\epsilon_0 = 0.8126\ldots$, and the root of equation (2.8) is $l_0 = 0.717$.

Thus, for Maclaurin ellipsoids with eccentricities $\epsilon > \epsilon_0$ the quadratic form of the variables p, q, ω_1, ω_2, α', β', γ', α, β, η, with which the series expansion of the function (2.4) begins, is positive definite.

Among the terms of higher order in the expression of the function V there are, however, terms which, in addition to the variables α, β, η, depend on the variable ξ as well.

Such terms are of the lowest order, as is easily seen in the following

$$\frac{2M}{5}(\alpha - \beta)^2\left[\xi^2 - 2\omega\xi\,\frac{\alpha + \beta}{a_0} - 2\xi\eta\right]$$

Adding and subtracting in the square brackets squared terms of α, β, η we obtain

$$\frac{2M}{5}(\alpha-\beta)^2\left\{\left[\xi^2-2\omega\xi\frac{\alpha+\beta}{a_0}-2\xi\eta+k(\alpha^2+\beta^2+\eta^2)\right]-k(\alpha^2+\beta^2+\eta^2)\right\}$$

If the constant k is chosen so that $k > 2\omega^2/a_0^2 + 1$, then the quadratic form of the variables ξ, η, α, β, which are in the square brackets, will be positive definite. And now it is obvious that the function (2.4) will be a positive definite function with respect to the variables α', β', γ', α, β, p, q, ω_1, ω_2, η in a sufficiently small neighborhood of the zero values of these variables, if the quadratic part of the function V is positive definite.

Consequently, the stability in the sense of Liapunov with respect to the variables a, b, a', b', c', p, q, ω_1, ω_2, ω_3 of Maclaurin ellipsoids with eccentricities $\epsilon < \epsilon_0$ have been proved for the condition that the initial disturbances satisfy the Dirichlet hypotheses.

From the stability with respect to the indicated variables, because of the existence of first integrals of the equations of the disturbed motion of the form V_4 = const, and also V_1 = const or V_2 = const, we can draw conclusions about the stability of Maclaurin ellipsoids with eccentricities $\epsilon < \epsilon_0$ with respect to the variables c and r as well.

We note that if an additional condition be imposed so that the form of the fluid always remained an ellipsoid of revolution, then all Maclaurin ellipsoids will be stable figures of the rotating fluid. Indeed, in this case it is necessary to put $\alpha = \beta$ and in place of the quadratic form of the variables α, β, η entering into the expression for the function V which was considered earlier we will have a quadratic form of the variables α and η of the form

$$2\left(A_{11}+A_{12}+2\mu\frac{\omega^2}{a_0^2}\right)\alpha^2+4(\mu-C_{20})\frac{\omega}{a_0}\alpha\eta+\mu\eta^2$$

The conditions of positive definiteness of this quadratic expression have the forms:

$$A_{11}+A_{12}+2\mu\frac{\omega^2}{a_0^2}>0, \qquad \left(A_{11}+A_{12}+4C_{20}\frac{\omega^2}{a^2}\right)\mu-2C_{20}{}^2\frac{\omega^2}{a_0^2}>0$$

and a choice of the positive constant μ is always possible so that these conditions will be satisfied if only the first of the inequalities (2.6) is fulfilled. The latter, as was established earlier, always is satisfied for Maclaurin ellipsoids, which also proves the stated assertion.

3. We will consider the particular solution of the equations of motion

$$a=a_0, \quad b=b_0, \quad c=c_0, \quad a'=b'=c'=0, \quad p=q=r=0$$
$$\omega_1=\omega_2=0, \quad \omega_3=\Omega, \quad \sigma=\sigma_0 \tag{3.1}$$

which describe the motion of a fluid with velocities

$$v_x = -\frac{2a_0^2\,\Omega}{a_0^2+b_0^2}\,y, \qquad v_y = \frac{2b_0^2\,\Omega}{a_0^2+b_0^2}\,x, \qquad v_z = 0 \tag{3.2}$$

moreover, the surface (1.2) remains stationary. Equations (1.15) in this case take the forms

$$\left(P - \frac{4a_0^2b_0^2}{(a_0^2+b_0^2)^2}\,\Omega^2\right)a_0^2 = \left(Q - \frac{4a_0^2b_0^2}{(a_0^2+b_0^2)^2}\,\Omega^2\right)b_0^2 = Rc_0^2 = 2\sigma_0$$

and transform into equations (2.2) if the notation

$$\omega^2 = \frac{4a_0^2b_0^2}{(a_0^2+b_0^2)^2}\,\Omega^2$$

is introduced.

Thus, we obtain [5] a series of Dedekind ellipsoids which are identical in external form with the series of Jacobi ellipsoids.

Obviously, for

$$\frac{2a_0^2b_0^2\Omega^2}{\pi f\rho\,(a_0^2+b_0^2)^2} = 0.1871$$

the semi-axes of the ellipsoid (1.2) are equal, the fluid moves as a single solid body rotating about the z-axis with angular velocity Ω, and the Dedekind ellipsoid turns into the bifurcated ellipsoid E which belongs simultaneously to the series of Maclaurin and Jacobi ellipsoids.

We will investigate the stability of the latter, supposing in the disturbed motion

$$a = a_0 + \alpha, \qquad b = a_0 + \beta, \qquad c = c_0 + \gamma, \qquad \omega_3 = \Omega + \eta$$

The equations of the disturbed motion permit integrals of the form (2.3), the first two of which after replacing γ in the first by α and β with the help of the integral $V_4 = 0$ can be written in the form

$$\begin{aligned}
V_1^* = {} & A_{10}(p^2+q^2) + A_{20}(\omega_1^2+\omega_2^2) + C_{20}\eta^2 + 2C_{20}\Omega\eta\left(1+\frac{\alpha+\beta}{a_0}\right) + \\
& + \frac{M}{5}\Big\{\alpha'^2+\beta'^2+\gamma'^2 + 4\Omega^2 a_0(\alpha+\beta) - \Omega^2(\alpha^2+\beta^2-4\alpha\beta) + \\
& + 2R_0\frac{c_0^2}{a_0^2}(\alpha^2+\beta^2)\Big\} + \left(\frac{\partial^2 W}{\partial a^2} + \frac{c^2}{a^2}\frac{\partial^2 W}{\partial c^2} - 2\frac{c}{a}\frac{\partial^2 W}{\partial c\,\partial a}\right)_0(\alpha^2+\beta^2) + \\
& + 2\left(\frac{\partial^2 W}{\partial a\,\partial b} + \frac{c^2}{a^2}\frac{\partial^2 W}{\partial c^2} - 2\frac{c}{a}\frac{\partial^2 W}{\partial c\,\partial a} + \frac{M}{5}R\frac{c^2}{a^2}\right)_0\alpha\beta + \ldots = \text{const}
\end{aligned}$$

$$\begin{aligned}
V_2 = {} & A_{10}^2(p^2+q^2) + 2A_{10}A_{20}(p\omega_1+q\omega_2) + A_{20}^2(\omega_1^2+\omega_2^2) + C_{20}^2\eta^2 + \\
& + \frac{2M}{5}C_{20}\Omega^2\left[2a_0(\alpha+\beta)+6\alpha\beta\right] + 2C_{20}\Omega\eta\left(C_{20} + 2C_{20}\frac{\alpha+\beta}{a_0}\right) + \ldots = \text{const}
\end{aligned}$$

and the two others keep the same form with ω replaced by Ω.

We will construct a function of the form

$$V = V_1^* - \frac{1}{C_{20}} V_2 + \mu \frac{c_0^4}{4\Omega^2} V_3^{*2} \tag{3.3}$$

which in terms of the smallest dimension differs from the function (2.4) in the replacement of ω by Ω and of the coefficients (2.5) by the following:

$$A_{11} = \left(\frac{\partial^2 W}{\partial a^2} + \frac{c^2}{a^2}\frac{\partial^2 W}{\partial c^2} - 2\frac{c}{a}\frac{\partial^2 W}{\partial c\,\partial a} + 2R\frac{c^2}{a^2}\frac{M}{5}\right)_0 - \frac{M}{5}\Omega^2$$

$$A_{12} = \left(\frac{\partial^2 W}{\partial a\,\partial b} + \frac{c^2}{a^2}\frac{\partial^2 W}{\partial c^2} - 2\frac{c}{a}\frac{\partial^2 W}{\partial c\,\partial a} + \frac{M}{5}R\frac{c^2}{a^2}\right)_0 - \frac{4M}{5}\Omega^2 \tag{3.4}$$

The quadratic part of the function (3.3) will be positive definite with respect to the variables α β, α', β', γ', p, q, ω_1, ω_2, η if the conditions (2.6) are fulfilled. Comparing the coefficients (2.5) and (3.4), we can convince ourselves that the latter are obtained from the former upon replacing ω by Ω and adding to the corresponding items $2/5\,M\Omega^2$ and $-2/5M\Omega^2$. Therefore, the first of the equations (2.6) keeps just the same form, which also is in $n^\circ 2$, and is satisfied, and the second in the case under consideration reduces to the form

$$\Omega^2 > 0$$

and is also satisfied. In the same way the stability of the bifurcated ellipsoid E under ellipsoidal disturbances with respect to the variables p, q, ω_1, ω_2, ω_3, a, b, c, a', b', c' can be proved to a first approximation.

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